CHAPTER 4

STATICALLY INDETERMINATE STRUCTURES

4.1.INTRODUCTION

A statically determinate structure can be analyzed for external reactions using the equilibrium equations and conditions that are discussed in statics. A structural system is termed statically indeterminate when there are more unknown external reactions than there are equilibrium equations and conditions of statics. The structure is statically indeterminate because of additional reaction constraints. These constraints are actually additional boundary conditions, which are used to formulate equations that combine with the equations of statics. The additional boundary conditions will be called deformation conditions. In any analysis the total number of unknown reactions is equal to the number of equations of statics plus the deformation conditions. The purpose of this chapter is to illustrate basic methods for interpreting the deformation conditions.

Example

1. As shown in Fig. 4.1a, a rigid horizontal bar is supported by a hinge at A and by two steel cables BD and CE, which are of equal length, L = 0.8 m, and cross-sectional area, A = 140 mm^2. Calculate the stress in each cable due to a force of 40 kN, applied as shown in the figure. Assume that the yielding stress is 250 MPa and that E = 200 GPa.

Solution

The free-body diagram of the bar given in Fig. 4.1b shows four unknown forces. Note that the rotated position of the bar after loading is indicated by the dashed lines in the figure.

Statics: Applying equations of equilibrium to Fig. 4.1b, we have

$$\begin{aligned} &\rightarrow \Sigma F_x = 0: \quad R_{Ax} = 0 \\ &\uparrow \Sigma F_y = 0: \quad -R_{Ay} + F_{BD} + F_{CE} - 40 = 0 \qquad (a) \\ &\circlearrowleft \Sigma M_A = 0: \quad F_{BD} + 2F_{CE} - 40\,(1.4) = 0 \end{aligned}$$

It is observed that the system is statically indeterminate to the first degree.

Deformations: The elongations of members BD and EC are given by

$$\delta_B = \frac{F_{BD}L}{AE} \downarrow \qquad \delta_C = \frac{F_{CE}L}{AE} \downarrow \qquad (b)$$

Geometry: The condition of geometric compatibility is determined by referring to Fig. 4.1b. From similar triangles,

$$\delta_C = 2\delta_B \qquad \text{(c)}$$

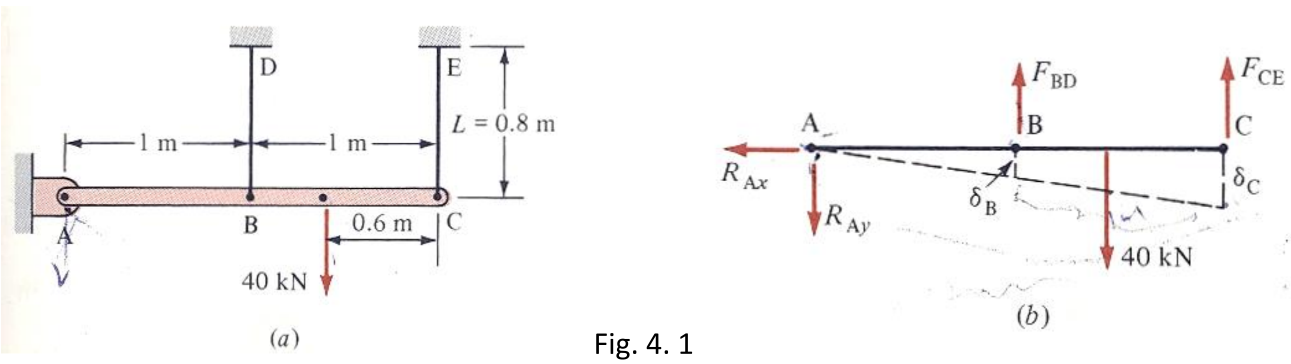


Fig. 4. 1

This, together with Eqs. (b), leads to

$$F_{CE} = 2F_{BD} \qquad \text{(d)}$$

Solving Eqs. (a) and (d), F_{BD}= 11.2 kN, F_{CE}= 22.4 kN, and R_{Ay} = 6.4 kN.Having found the axial forces, we find the deformations of the cables, from Eqs. (b):

$$\delta_B = \frac{(11.2 \times 10^3)(0.8)}{140 \times 10^{-6}(200 \times 10^9)} = 0.32 \times 10^{-3}\ \text{m} = 0.32\ \text{mm} \downarrow$$

$$\delta_C = 0.64\ \text{mm} \downarrow$$

The stresses in the members are

$$\sigma_{BD} = \frac{11.2 \times 10^3}{140 \times 10^{-6}} = 80\ \text{MPa} \qquad \sigma_{CE} = 160\ \text{MPa}$$

Since the stresses are below the yielding strength of the material, the solution is acceptable. If either of these stresses were above the yield point, the results would not be valid and a redesign would be required.

4.2. APPLICATION OF THE METHOD OF SUPERPOSITION

For many statically indeterminate situations, the principle of superposition offers an effective approach to solution. Consider, for example, the bar of Fig. 4.2a, replaced by the bars shown in Fig. 4.2c and d. At point B, the bar now experiences the displacements δ_p, and δ_R due, respectively, to P and R_B. Superposition of these displacements, in order that there be no movement of the right end of the bar, yields

$$\delta_P + \delta_R = 0$$

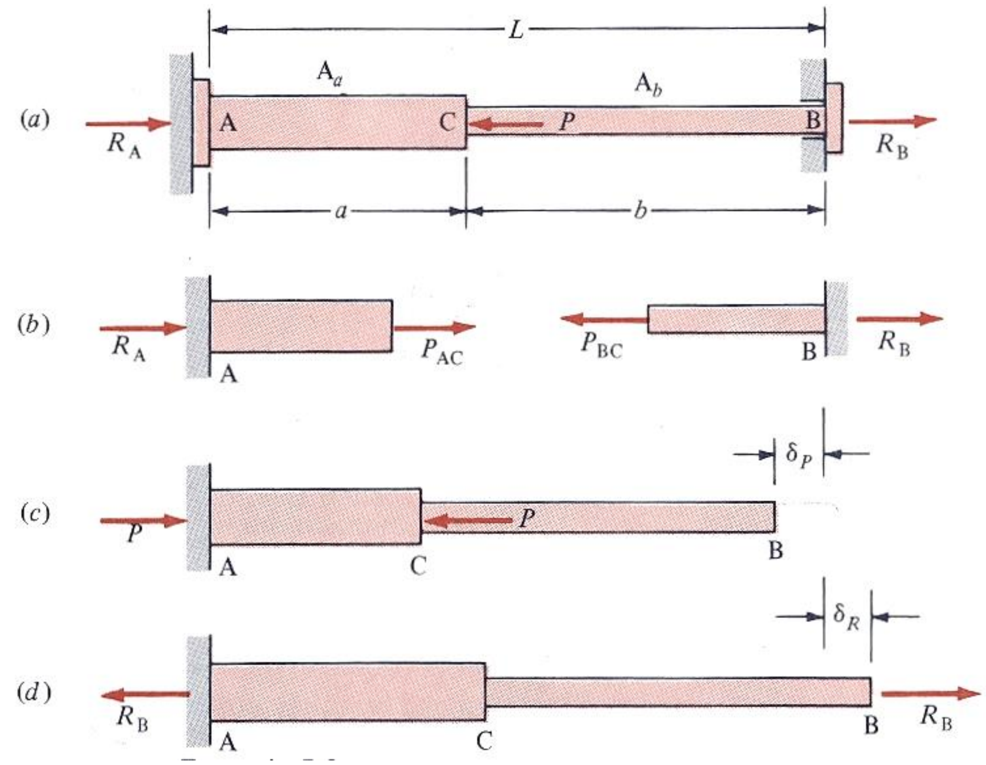


Fig.4.2

Applying equation below and taking the elongations to be positive, we have

$$-\frac{Pa}{A_aE} + \left(\frac{R_Ba}{A_aE} + \frac{R_Bb}{A_bE}\right) = 0 \qquad (a)$$

from which

$$R_B = \frac{P}{1 + (bA_a/aA_b)}$$

The remaining reactioncan be obtained from the condition of statics:

$R_A + R_B = P$

The method of superposition employed above may be summarized as follows:

1. One of the unknown reactions is designated as redundant and released from the member by removing the support.

2. The remaining member, which is rendered statically determinate, is loaded by the actual load (P) and the redundant (R_B) itself. Note thatthe redundant is considered to be an unknown load.

3. The expressions for the displacements due to these loads are obtained and substituted into the equation of geometric compatibility to calculate the redundant reaction. The other unknown reaction is found by applying statics.

In problems involving a large degree of indeterminacy, the principle of superposition may be used to good advantage for simplification of analysis. Since this approach employs the redundant reactions as the unknowns, it is often referred to as the**forcemethod**. Another name is the **flexibility method**, because flexibilities appear in the compatibility requirement. Clearly, once the redundant reaction is determined, the calculations of stress and deformation proceed in the usual manner.

4.3. THERMAL DEFORMATION AND STRESS

Consider the consequences of increasing or decreasing the uniform temperature of an unconstrained isotropic body. The resultant expansion or contraction occurs in such a way as to cause a cubic element of the solid to remain cubic while undergoing changes of length on each of its sides. Normal strains occur in all directions, unaccompanied by normal stresses. In addition, there are neither shearing strains nor shearing stresses.

The strain owing to a 1^0 temperature change is denoted by α and is called the **coefficient of thermal expansion**.

Thermal strain caused by a uniform increase in temperature ΔT is therefore

$$\varepsilon_t = \alpha \, \Delta T \qquad \text{...(1)}$$

The coefficient of expansion is approximately constant over a moderate range of temperature change. It represents a quantity per degree Celsius (l/°C), or per degree Fahrenheit (1/°F).

The free expansion or contraction owing to a change in temperature is readily found by applying Eq. (1). In an elastic body, thermal deformation caused by a uniform temperature change is

$$\delta_T = \alpha(\Delta T)L \qquad \text{...(2)}$$

where the L is the dimension of the body. The thermal strain and deformation

are positive if the temperature increases and negative if it decreases.

Stresses due to the restriction of thermal expansion or contraction of a body are called thermal stresses. In a statically determinate structure, a uniform temperature change will not cause any stresses, as deformations are permitted to occur freely. On the other hand, a temperature change in a structure that is supported in a statically indeterminate manner will induce stresses in the member.

The following example demonstrate the approach.

Example

2. As shown in Fig. 4.3a, a 30 mm diameter bronze cylinder is secured between a rigid cap and slab tightening two 20 mm diameter steel bolts. At 20°C, no deformation and stress exist in the assembly. Determine the stress in the bronze and steel at 70°C. Use E_s = 200 GPa, E_b = 83 GPa, α_s = 11.7 X 10^{-6}/°C, and α_b= 18.9 x 10^{-6}/°C.

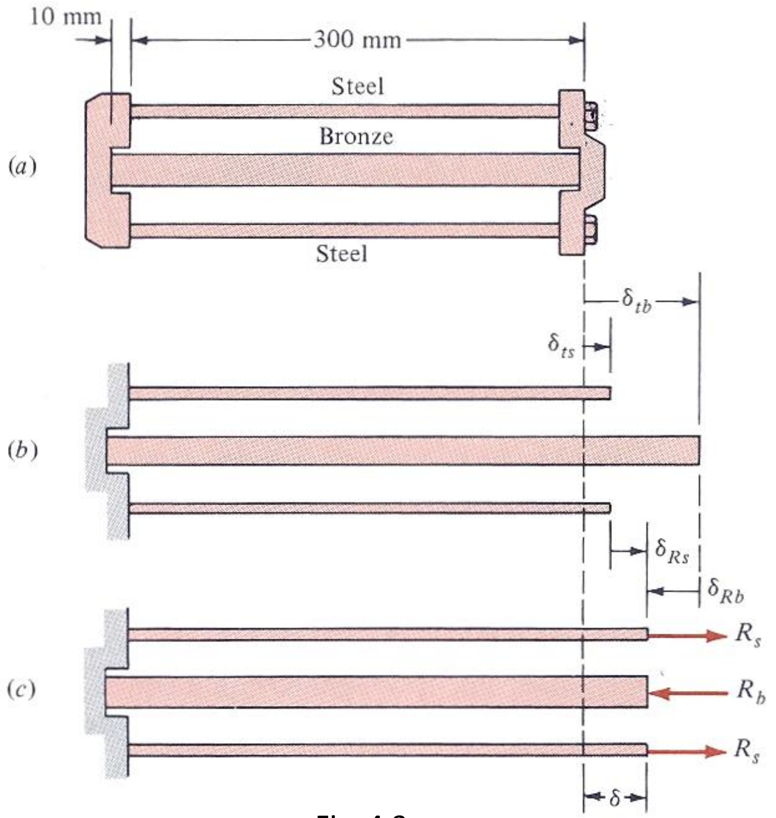


Fig. 4.3

Solution

After the temperature is raised by ΔT = 70 - 20 = 50°C, the cap and slab exerts equal and opposite forces R_b in the bronze cylinder and R_s in each steel bolt. With the cap detached (Fig. 4 .3b), thermal elongations $\delta_{\tau b}$, and $\delta_{\tau s}$, take place in the bronze cylinder and steel bolts, respectively. It is necessary, however, that the net deflection of these two members be the same. Thus the cylinder stretches the bolts, and the bolts in turn compress the cylinder until a final equilibrium position is reached (Fig. 4.3c).

Statics: The only equation of equilibrium for the system leads to

$$R_b = 2R_s \qquad (a)$$

containing both unknown reactions.

Deformations:

$$\delta_{tb} = 18.9 \times 10^{-6}(50)(0.31) = 0.293 \times 10^{-3} \text{ m}$$
$$\delta_{ts} = 11.7 \times 10^{-6}(50)(0.3) = 0.176 \times 10^{-3} \text{ m} \qquad (b)$$

and

$$\delta_{Rb} = \frac{R_b(0.31)}{\pi(0.015^2)(83 \times 10^9)} = 5.28(10^{-9})R_b$$
$$\delta_{Rs} = \frac{R_s(0.3)}{\pi(0.01^2)(200 \times 10^9)} = 4.77(10^{-9})R_s \qquad (c)$$

Geometry: Referring to the figure, we see that the deformations are related by

$$\delta = \delta_{tb} - \delta_{ts} = \delta_{Rb} + \delta_{Rs}$$

substituting Eqs. (b) and (c) into the above and solving the resulting expression and Eq. (a) yields

$$R_b = 15.26 \text{ kN} \qquad R_s = 7.63 \text{ kN}$$

Thus

$$\sigma_b = \frac{15.26 \times 10^3}{\pi(0.015^2)} = 21.59 \text{ MPa} \qquad \sigma_s = \frac{7.63 \times 10^3}{\pi(0.01^2)} = 24.29 \text{ MPa}$$

are the compressive stress in the cylinder and the tensile stress in each bolt, respectively.

4.4. STATICALLY INDETERMINATE STRUCTURES BY CASTIGLINO'S THEOREM

The reactions at the supports of a statically indeterminate elasticstructure may be determined by Castigliano's theorem. In the case of a structure indeterminate to the first degree, for example, we designateone of the reactions as redundant and eliminate or modify accordingly the corresponding support. The redundant reaction is then treated as an unknown load which, together with the other loads, must produce deformations which are compatible with the original supports. We first calculate the strain energy U of the structure due to the combined action of the given loads and the redundant reaction. Observing that the partial derivative of U with respect to the redundant reaction represents the deflection (or slope) at the support which has been eliminated or modified, we then set this derivative equal to zero and solve the equation obtained for the redundant reaction. The remaining reactions maybe obtained from the equations of statics.

***Note**: This is in the case of a rigid support allowing no deflection. For other types of support, the partial derivative of U should be set equal to the allowed deflection.*

<u>Example</u>

1. Determine the reactions at the supports for the prismatic beam and loading shown (Fig. 4.4).

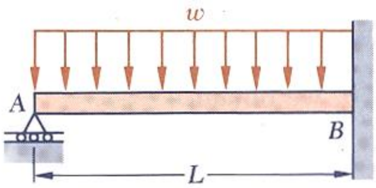


Fig. 4.4

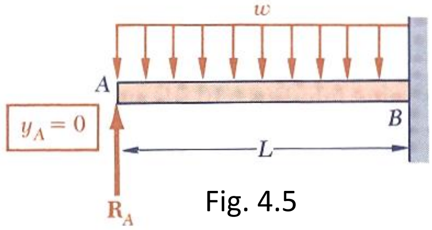


Fig. 4.5

The beam is statically indeterminate to the first degree.We consider the reaction at A as redundant and release the beam from that support. The reaction R_A is now considered as an unknown load (Fig. 4.5) and will be determined from the condition that the deflection y_A at A must be zero. By Castigliano's theorem $y_A = \partial U/\partial R_A$, where U is the strain energyof the beam under the distributed load and the redundant reaction. We know that

$$y_A = \frac{\partial U}{\partial R_A} = \int_0^L \frac{M}{EI}\frac{\partial M}{\partial R_A}\,dx \qquad \text{...(1)}$$

We now express the bending moment M for the loading of Fig. 4.5.

The bending moment at a distance x from A is

$$M = R_A x - \tfrac{1}{2}wx^2 \qquad \text{...(2)}$$

and its derivative with respect to R_A is

$$\frac{\partial M}{\partial R_A} = x \qquad \text{...(3)}$$

Substituting for M and $\partial U/\partial R_A$ from (2) and (3) into (1), we write

$$y_A = \frac{1}{EI}\int_0^L \left(R_A x^2 - \frac{1}{2}wx^3\right) dx = \frac{1}{EI}\left(\frac{R_A L^3}{3} - \frac{wL^4}{8}\right)$$

Setting $y_A = 0$ and solving for R_A, we have

$$R_A = \tfrac{3}{8}wL \qquad \mathbf{R}_A = \tfrac{3}{8}wL \uparrow$$

From the conditions of equilibrium for the beam, we find that the reaction at B consists of the following force and couple:

$$\mathbf{R}_B = \tfrac{5}{8}wL \uparrow \qquad \mathbf{M}_B = \tfrac{1}{8}wL^2 \downarrow$$

2. A load P is supported at B by three rods of the same material and the same cross-sectional area A (Fig. 4.6). Determine the force in each rod.

The structure is statically indeterminate to the first degree. We consider the reaction at H as redundant and release rod BH from its support at H. The reaction R_H is now considered as an unknown load (Fig. 4.7) and will be determined from the condition that the deflection y_H of point H must be zero. By Castigliano's theorem $y_H = \partial U/\partial R_H$, where U is the strain energy of the three rod system under the load P and the redundant reaction R_H. We know that

$$x_j = \frac{\partial U}{\partial P_j} = \sum_{i=1}^{n} \frac{F_i L_i}{A_i E} \frac{\partial F_i}{\partial P_j} \qquad \text{...(4)}$$

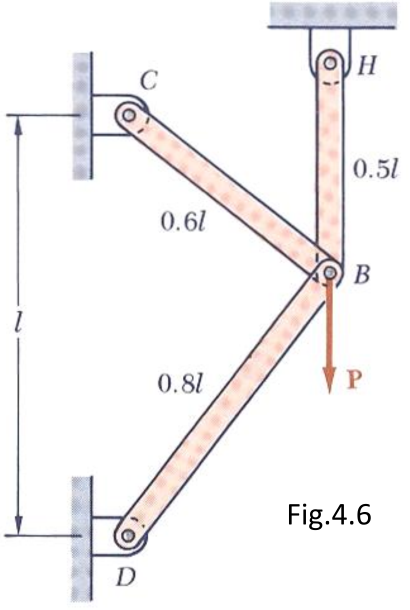


Fig.4.6

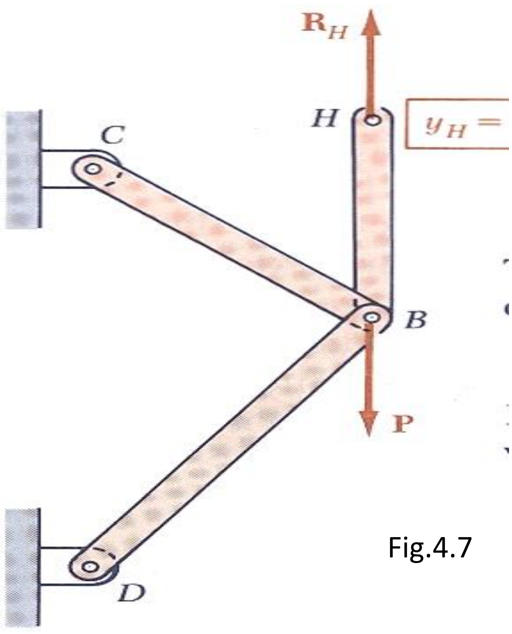
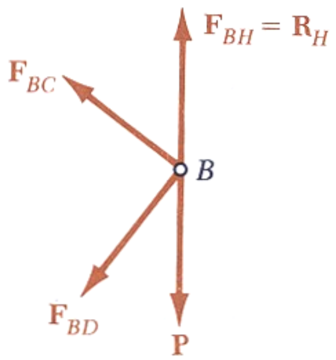


Fig.4.7

$$y_H = \frac{F_{BC}(BC)}{AE}\frac{\partial F_{BC}}{\partial R_H} + \frac{F_{BD}(BD)}{AE}\frac{\partial F_{BD}}{\partial R_H} + \frac{F_{BH}(BH)}{AE}\frac{\partial F_{BH}}{\partial R_H}$$

∴

We note that the force in rod BH is equal to R_H and write



$$F_{BH} = R_H \qquad ...(6)$$

Then, from the free-body diagram of pin B (Fig. 4.8), weobtain

$$F_{BC} = 0.6P - 0.6R_H \qquad F_{BD} = 0.8R_H - 0.8P \qquad ...(7)$$

Differentiating with respect to R_H the force in each rod, we write

$$\frac{\partial F_{BC}}{\partial R_H} = -0.6 \qquad \frac{\partial F_{BD}}{\partial R_H} = 0.8 \qquad \frac{\partial F_{BH}}{\partial R_H} = 1 \qquad ...(8)$$

to (1 ... ng that the lengths BC, DV, an

Substituting from (6), (7), and (8) into (5), and noting that the lengths BC, BD, and BH are respectively equal to $0.6l$, $0.8l$, and $0.5l$, we write

$$y_H = \frac{1}{AE}[(0.6P - 0.6R_H)(0.6l)(-0.6) + (0.8R_H - 0.8P)(0.8l)(0.8) + R_H(0.5l)(1)]$$

Setting $y_H = 0$, we obtain

$$1.228R_H - 0.728P = 0$$

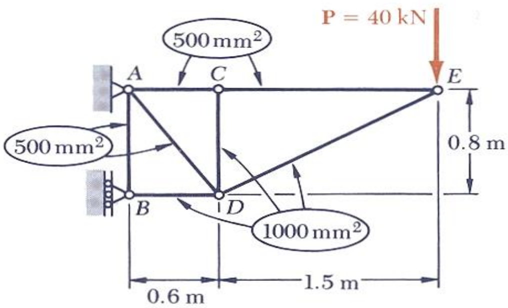
and, solving for RH,

$$R_H = 0.593P$$

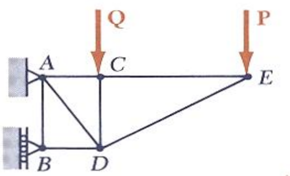
Carrying this value into Eqs. (6) and (7), we obtain the forces in the three rods:

$$F_{BC} = +0.244P \qquad F_{BD} = -0.326P \qquad F_{BH} = +0.593P$$

3.For the truss and loading shown in figure below, determine the vertical deflection of joint C.

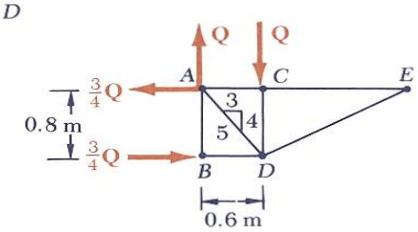


Castigliano's Theorem. Since no vertical load is applied at joint C, we introduce the dummy load Q as shown. Using Castigliano's theorem, we have, since E = constant,



$$y_C = \sum \left(\frac{F_i L_i}{A_i E}\right) \frac{\partial F_i}{\partial Q} = \frac{1}{E} \sum \left(\frac{F_i L_i}{A_i}\right) \frac{\partial F_i}{\partial Q} \qquad ...(1)$$

where F, is the force in a given member under the combined loading of P and Q.

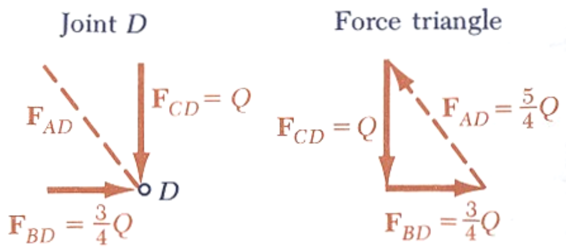


Force in Members. Considering in sequence the equilibrium ofjoints E, C, B, and D, we determine the force in each member caused by the load Q.

Joint E: $F_{CE} = F_{DE} = 0$

Joint C: $F_{AC} = 0$; $F_{CD} = -Q$

Joint B: $F_{AB} = 0$; $F_{BD} = -\frac{3}{4}Q$



The force in each member caused by the load P was previously found in Sample Problem of previous chapter. The total force in each member under the combined action of Q and P is shown in the following table. Forming $\partial F_i / \partial Q$ for each member, we then compute $(F_i L_i / A_i)(\partial F_i / \partial Q)$ as indicated in the table.

Member	F_i	$\partial F_i/\partial Q$	L_i, m	A_i, m²	$\left(\frac{F_i L_i}{A_i}\right)\frac{\partial F_i}{\partial Q}$
AB	0	0	0.8	500×10^{-6}	0
AC	$+15P/8$	0	0.6	500×10^{-6}	0
AD	$+5P/4 + 5Q/4$	$\frac{5}{4}$	1.0	500×10^{-6}	$+3125P + 3125Q$
BD	$-21P/8 - 3Q/4$	$-\frac{3}{4}$	0.6	1000×10^{-6}	$+1181P + 338Q$
CD	$-Q$	-1	0.8	1000×10^{-6}	$+ 800Q$
CE	$+15P/8$	0	1.5	500×10^{-6}	0
DE	$-17P/8$	0	1.7	1000×10^{-6}	0

$$\sum\left(\frac{F_i L_i}{A_i}\right)\frac{\partial F_i}{\partial Q} = 4306P + 4263Q$$

Deflection of C. Substituting into Eq. (1), we have

$$y_C = \frac{1}{E}\sum\left(\frac{F_i L_i}{A_i}\right)\frac{\partial F_i}{\partial Q} = \frac{1}{E}(4306P + 4263Q)$$

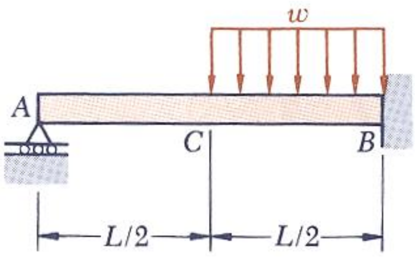
Since the load Q is not part of the original loading, we set Q = 0. Substituting the given data, P = 40 kN and E = 70 GPa, we find

$$y_C = \frac{4306(40 \times 10^3 \text{ N})}{70 \times 10^9 \text{ Pa}} = 2.46 \times 10^{-3} \text{ m} \qquad y_C = 2.46 \text{ mm} \downarrow \blacktriangleleft$$

EXERCISES

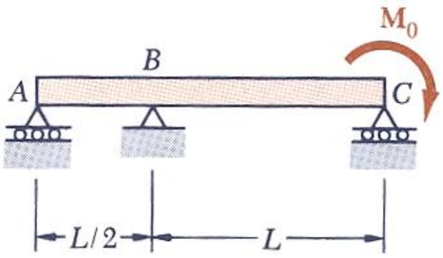
1. Determine the reaction at the roller support and draw the bending-moment diagram for the beam and loading shown.

[Ans. R$_A$= $7wL/128\uparrow; M_B = -9wL/128$**]**



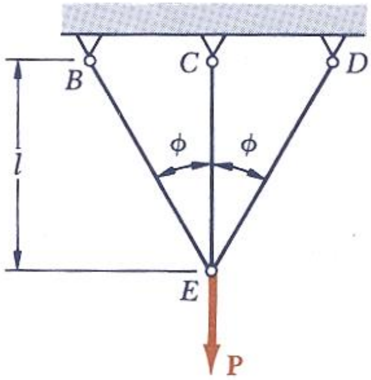
2. For the uniform beam and loading shown, determine the reaction at each support.

[Ans. R$_A$= $2M_o/3L\uparrow; R_B = 2M_o/L\downarrow; R_c = 4M_o/3L\uparrow$**]**



3. Three members of the same material and same cross- sectional area are used to support the load P. Determine the force in each member when $\phi = 30°$.

[Ans. F$_{BE}$ = FDE = 0.326P; F$_{CE}$ = 0.435P]



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